

Dielectric properties of interacting storage ring plasmas

A. Selchow, K. Morawetz

Fachbereich Physik, Universität Rostock, D-18051 Rostock, Germany

(February 2, 2008)

A dielectric function including collisional correlations is derived by linearizing the self consistent Vlasov equation with a Fokker-Planck collision integral. The calculation yields the same type of dielectric function as in the standard-theory of Schottky-noise in storage rings. This dielectric function is compared with the Mermin-dielectric function derived from a kinetic equation with relaxation time approximation. We observe that these functions are identical, however the Mermin-DF is computationally advantageous. The limits of both dielectric functions are given and the sum rules are proven. We apply these dielectric functions for typical storage ring plasmas and calculate the stopping power and the plasmon excitation spectrum.

I. INTRODUCTION

During the last ten years experiments with ions (p^+ up to U^{92+}) in storage rings gained importance in the field of spectroscopy and plasma physics. The stored and cooled ion beams have a high luminosity for recombination experiments and inertial confined fusion investigations. In particular it is of basic interest to study the transition between the weak and strong coupled plasma or even the transition to a crystalline state of a cooled ion beam [1]. The most important prerequisite for obtaining dense states is strong electron and laser cooling. The electron cooling force can be described as stopping power acting on an ion beam in an electron plasma [2]. Another plasma phenomena in dense beams are collective excitations (plasmons, shear modes) which are detectable by the Schottky noise [3]. All items - the pair distribution function of a state, the stopping power and the shape of the collective excitations are related to the dielectric function $\epsilon(\vec{q}, \omega)$.

Within the linear response theory the polarizability $\Pi(\vec{q}, \omega)$ (and altogether the dielectric function $\epsilon(\vec{q}, \omega)$) is defined by the variation of particle density $\delta n(\vec{q}, \omega)$ in reaction to an external field $\delta U_{ext}(\vec{q}, \omega)$ via

$$\Pi(\vec{q}, \omega) = \frac{\delta n(\vec{q}, \omega)}{\delta U_{ext}}. \quad (1)$$

The connection to the dielectric function (DF) is given by

$$\epsilon(\vec{q}, \omega) = 1 + V_C \Pi(\vec{q}, \omega). \quad (2)$$

The captured ions in storage rings are moving in front of a background of the confining fields ensuring approximately the charge neutrality in the system. This nearly neutral system of ions interacting via the Coulomb potential V_C immersed in a homogeneous background of opposite charge is usually called an one-component-plasma (OCP).

An unpleasant problem is the temperature anisotropy. The longitudinal temperature (T_{\parallel}) differs from the transversal (T_{\perp}) (referring to the beam-axis) because only the longitudinal direction is cooled directly. Even taking into account relaxation processes between the degrees of freedom and possible transversal cooling the temperature difference maintains.

In this paper we focus on experiments done by the Heidelberg TSR group with $^9\text{Be}^+$ and $^{12}\text{C}^{6+}$ ions cooled by an electron beam [1,4]. The Be-ions can be cooled further down to few mK by applying laser cooling.

For estimations about the ideality and collision-numbers we employ the longitudinal temperature T_{\parallel} . The density of the ion beam can be calculated using the current j of the beam profile (the diameter) measured by the beam profile monitors x_{BPM} and the value of the betatron function on this position β_{BPM}

$$n = \frac{jQ\beta_{BPM}}{2\pi Z x_{BPM}^2 R v_0}. \quad (3)$$

Here denotes v_0 the ion beam velocity, R is the ring radius and Z is the charge state of the ions. The transversal tune Q amounts to 2.8.

Parameter	Be ⁺	C ⁶⁺	e ⁻
n	$2.3 \cdot 10^{13} \text{m}^{-3}$	$1.6 \cdot 10^{13} \text{m}^{-3}$	$2.9 \cdot 10^{13} \text{m}^{-3}$
T_{\parallel}	6K	8000K	3.5K
Γ	0.13	0.0031	0.23
λ	$1.7 \cdot 10^7 \text{s}^{-1}$	$4.0 \cdot 10^5 \text{s}^{-1}$	$3.4 \cdot 10^{12} \text{s}^{-1}$
ω_{pl}	$2.1 \cdot 10^6 \text{s}^{-1}$	$9.2 \cdot 10^6 \text{s}^{-1}$	$3.0 \cdot 10^8 \text{s}^{-1}$

TABLE I. Parameters for several experiments in the TSR. The explanations are given in the text. Data are from [1,4].

The essential parameter for characterizing the coupling is the nonideality or plasma parameter

$$\Gamma = \frac{e^2}{4\pi\epsilon_0 k_B T} \left(\frac{4\pi n}{3} \right)^{\frac{1}{3}} \quad (4)$$

which is the ratio of potential and thermal energy. Further essential quantities are the plasma-frequency

$$\omega_{pl} = \sqrt{\frac{ne^2}{\epsilon_0 m}} \quad (5)$$

and the inverse screening length

$$\kappa = \sqrt{\frac{ne^2}{\epsilon_0 k_B T}}. \quad (6)$$

Another important parameter characterizing the plasma is the collision frequency, or friction coefficient λ which is the inverse relaxation time

$$\lambda = \frac{1}{\tau} = n\sigma(v_{th})v_{th} \quad (7)$$

$$\sigma(v) = 4\pi \left(\frac{2e^2}{12\epsilon_0 k_B T} \right) \Lambda(v).$$

Here is $\sigma(v_{th})$ the cross-section at thermal velocity $v_{th}^2 = 2k_B T/m$ for ion-ion collisions, $\Lambda(v)$ denotes the Coulomb logarithm e.g. in Brooks-Herring approximation.

The collisions between the ions play an essential role in this storage-ring-plasmas being responsible for effects like intra-beam scattering (IBS) [5]. That means an expansion of the (ion) beam due to ion-ion collisions has to be suppressed by electron cooling. Consequently for a sufficient description a dielectric function $\epsilon(\vec{q}, \omega)$ including these collisions should be considered [6]. In this paper two practical ways will be shown in chapter II.

An essential property of every dielectric function is the fulfillment of the sum rules. The most strongest are the longitudinal frequency sum rule

$$\int_0^\infty \frac{2\omega}{\pi\omega_{pl}^2} \text{Im}\epsilon^{-1}(\vec{q}, \omega) d\omega = 1 \quad (8)$$

and the conductivity sum rule

$$\int_0^\infty \frac{2\omega}{\pi\omega_{pl}^2} \text{Im}\epsilon(\vec{q}, \omega) d\omega = -1, \quad (9)$$

moreover the compressibility sum rule

$$\lim_{q \rightarrow 0} \int_0^\infty \frac{2}{\pi\omega_{pl}^2} \frac{1}{\omega} \text{Im}\epsilon^{-1}(\vec{q}, \omega) d\omega = 1 \quad (10)$$

and the perfect screening sum rule

$$\lim_{q \rightarrow 0} \int_0^\infty \frac{2}{\pi\omega_{pl}^2} \frac{1}{\omega} \text{Im}\epsilon(\vec{q}, \omega) d\omega = -1. \quad (11)$$

The validity of these sum rules is an essential statement about the quality and reliability of the dielectric function.

The outline of the paper is as follows: In Chapter II A we give a short rederivation of Mermin DF and the DF from a Fokker-Planck equation is given in Chapter II B. In Chapter II C we compare both DF's and the sum rules are proven in Chapter II D. In Chapter III A we present the application to the stopping power and in Chapter III B we calculate the Schottky spectra.

II. DIELECTRIC FUNCTIONS WITH CORRELATIONS

A. Mermin-dielectric function

For calculating a dielectric function including collisions between the particles with mass m , N. D. Mermin [7] suggested a particle number conserving dielectric function. We shortly sketch his derivation for the classical case starting with a kinetic equation in relaxation time approximation

$$\frac{\partial}{\partial t} f(\vec{r}, \vec{v}, t) + \vec{v} \frac{\partial}{\partial \vec{r}} f(\vec{r}, \vec{v}, t) + \frac{\partial}{\partial \vec{r}} \frac{U(\vec{r}, t)}{m} \frac{\partial}{\partial \vec{v}} f(\vec{r}, \vec{v}, t) = - \frac{f(\vec{r}, \vec{v}, t) - f_0(\vec{v})}{\tau}. \quad (12)$$

This kinetic equation describes the development of a particle distribution function $f(\vec{r}, \vec{v}, t)$ consisting of an equilibrium part $f_0(\vec{v})$ and a non-equilibrium part $\delta f(\vec{r}, \vec{v}, t)$

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{v}) + \delta f(\vec{r}, \vec{v}, t). \quad (13)$$

The mean field $U(\vec{r}, t)$ is composed of an external part and a part arising from the induced particle density δn

$$U(\vec{q}, \omega) = \delta U_{ext} + V \delta n(\vec{q}, \omega). \quad (14)$$

One gets the induced particle density δn by linearization of (12) and integrating the solution of δf over the velocity \vec{v} . After Fourier transformation $t \rightarrow \omega$ and $r \rightarrow q$ the following polarization function $\Pi(\vec{q}, \omega)$ is obtained

$$\begin{aligned} \delta n(\vec{q}, \omega) &= \int \delta f(\vec{q}, \vec{v}, \omega) d^3v \\ &= \frac{\Pi_0(\vec{q}, \omega + \frac{i}{\tau})}{1 - V(\vec{q})\Pi_0(\vec{q}, \omega + \frac{i}{\tau})} \delta U_{ext}(\vec{q}, \omega) \end{aligned} \quad (15)$$

with the RPA or Lindhard polarization function

$$\Pi_0(\vec{q}, \omega) = \int d^3v \frac{\frac{\vec{q}}{m} \frac{\partial}{\partial \vec{v}} f(\vec{v})}{\vec{v} \cdot \vec{q} - \omega + i/\tau}. \quad (16)$$

The RPA dielectric function (2) in classical limit reads

$$\begin{aligned} \epsilon(q, \omega) &= 1 + \frac{\kappa^2}{q^2} \left(1 - 2x_c e^{-x_c^2} \int_0^{x_c} e^{t^2} dt + i\sqrt{\pi} x_c e^{-x_c^2} \right) \\ x_c &= \sqrt{\frac{m}{2k_B T}} \frac{\omega}{q} \end{aligned} \quad (17)$$

and fulfills all sum rules (8)-(11). Shifting the frequency into the complex plane according to 16 one gets the relaxation dielectric function. This expression does not fulfill the limit of static screening and has a non Drude-like high frequency behavior which leads to a violation of the sum rules (8)-(11).

In [7] was suggested a more sophisticated dielectric function by considering the relaxation ansatz

$$\frac{\partial}{\partial t} f(\vec{r}, \vec{v}, t) + \vec{v} \frac{\partial}{\partial \vec{r}} f(\vec{r}, \vec{v}, t) + \frac{\partial}{\partial \vec{r}} \frac{U(\vec{r}, t)}{m} \frac{\partial}{\partial \vec{v}} f(\vec{r}, \vec{v}, t) = - \frac{f(\vec{r}, \vec{v}, t) - \tilde{f}_0(\vec{r}, \vec{v}, t)}{\tau} . \quad (18)$$

with respect to a local equilibrium distribution function

$$\tilde{f}_0(\vec{r}, \vec{v}, t) = \exp \left[-\frac{mv^2}{2k_B T} + \frac{\mu + \delta\mu(\vec{r}, t)}{k_B T} \right] \quad (19)$$

instead of the global distribution $f_0(\vec{v})$ in (12). In the simplest case one can specify the local distribution by a small fluctuation in the chemical potential $\delta\mu$ related to the density-fluctuation δn .

The Mermin-dielectric function is derived by solving (18) using an expansion of the local equilibrium distribution function in powers of $\delta\mu$

$$\tilde{f}_0(\vec{q}, \vec{v}, \omega) = f_0(\vec{v}) - \frac{\vec{q} \vec{v} f_0(\vec{v})}{m \vec{q} \vec{v}} \delta\mu(\vec{q}, \omega). \quad (20)$$

$\delta\mu$ is determined by the particle number conservation $\omega \delta n(\vec{q}, \omega) = \int \vec{v} \vec{q} \delta f(\vec{q}, \vec{v}, \omega) d^3 v$. leading to [7]

$$\delta\mu(\vec{q}, \omega) = \frac{\delta n(\vec{q}, \omega)}{\Pi(\vec{q}, 0)}. \quad (21)$$

Finally one obtains from (18) and (20) for the polarization function

$$\Pi_M(\vec{q}, \omega) = \frac{\Pi_0(\vec{q}, \omega + \frac{i}{\tau})}{1 - \frac{1}{1 - i\omega\tau} \left(1 - \frac{\Pi_0(\vec{q}, \omega + \frac{i}{\tau})}{\Pi_0(\vec{q}, 0)} \right)}. \quad (22)$$

Instead of (15) we arrive at a density variation

$$\delta n(\vec{q}, \omega) = \frac{\Pi_M(\vec{q}, \omega + \frac{i}{\tau})}{1 - V(\vec{q}) \Pi_M(\vec{q}, \omega + \frac{i}{\tau})} \delta U_{ext}(\vec{q}, \omega) \quad (23)$$

and with (2) the Mermin-dielectric function finally has the shape

$$\epsilon_M \left(\vec{q}, \omega + \frac{i}{\tau} \right) = 1 + \frac{\left(1 + \frac{i}{\omega\tau} \right) (\epsilon(\vec{q}, \omega + \frac{i}{\tau}) - 1)}{1 + \frac{i}{\omega\tau} \frac{\epsilon(\vec{q}, \omega + \frac{i}{\tau}) - 1}{\epsilon(\vec{q}, 0) - 1}}. \quad (24)$$

Here denotes $\epsilon(\vec{q}, \omega + \frac{i}{\tau})$ the dielectric function (17) in relaxation time approximation. It is easy to see, that in the limit $\tau \rightarrow \infty$ the Mermin-dielectric function reproduces the RPA-dielectric function (17).

B. The Vlasov-Fokker-Planck Equation

Now we examine another kinetic equation - the Vlasov equation with the Fokker-Planck collision integral which has been used to predict the Schottky noise of an ion beam [3]

$$\begin{aligned} \frac{\partial}{\partial t} f(\vec{r}, \vec{v}, t) + \vec{v} \frac{\partial}{\partial \vec{r}} f(\vec{r}, \vec{v}, t) + \frac{\partial}{\partial \vec{r}} \frac{U(\vec{r}, t)}{m} \frac{\partial}{\partial \vec{v}} f(\vec{r}, \vec{v}, t) \\ = \lambda \frac{\partial}{\partial \vec{v}} \left(\frac{D}{\lambda} \frac{\partial}{\partial \vec{v}} + \vec{v} \right) f(\vec{r}, \vec{v}, t). \end{aligned} \quad (25)$$

The application of the Fokker-Planck collision term is valid for weak collisions (it means low q -values) because it represents an expansion of the collision integral in momentum space. With the collision integral of the Fokker-Planck equation one includes the fluctuations of the distribution function due to collisions. It describes the balance between dynamical friction $\lambda \frac{\partial}{\partial \vec{v}} (f(\vec{r}, \vec{v}, t))$ holding the velocity-distribution sharply near zero-velocity and the diffusion $D \frac{\partial^2}{\partial v^2} f(\vec{r}, \vec{v}, t)$ flattening the velocity distribution. The coefficients λ and D in the Fokker-Planck equation are related by the Einstein relation

$$\frac{D}{\lambda} = \frac{k_B T}{m}. \quad (26)$$

As already mentioned above the friction coefficient λ is equal to the inverse relaxation time. Obviously, the drift coefficient $\lambda \vec{v}$ is linear in the velocity as long as the diffusion coefficient D is a constant. The Fokker-Planck collision term ensures the particle conservation. Due to the Einstein relation we have a proper balance between friction and diffusion. So we expect that similar physics is included as the Mermin extension of the simple relaxation time approximation in the last paragraph.

We solve this Fokker-Planck equation again within the linear response. A sketch of the derivation can be found in appendix A with the result for the dielectric function

$$\begin{aligned} \epsilon_{VFP}(q, \omega) = 1 + \frac{\kappa^2}{q^2} \\ \times \left(1 + \frac{i\omega}{\frac{k_B T}{m} \frac{q^2}{\lambda} - i\omega} {}_1F_1 \left[1, 1 + \frac{k_B T}{m \lambda^2} q^2 - i \frac{\omega}{\lambda}; \frac{k_B T}{m \lambda^2} q^2 \right] \right) \end{aligned} \quad (27)$$

and ${}_1F_1$ denotes the confluent hypergeometric function. This dielectric function has been given in [3] and is valid for an isotropic plasma in three dimensions.

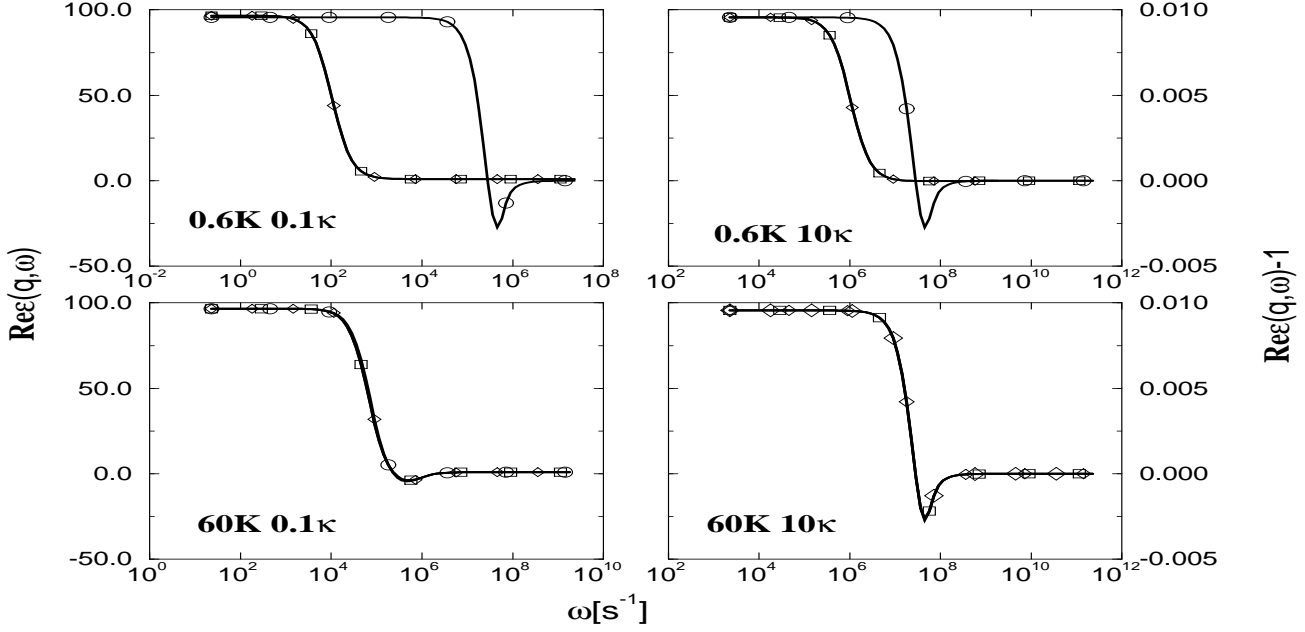


FIG. 1. Comparison of the real parts of the RPA- (circles), Mermin- (squares) and VFP-DF (diamonds). We have chosen temperatures of $0.6 \cdots 60\text{K}$ available in the longitudinal direction of an ion beam and wave numbers below and above the inverse Debye length κ (6). The particle density is $n = 2.3 \cdot 10^{13}\text{m}^{-3}$ of single charged Beryllium ions. The real parts of Mermin- and the VFP-DF are identical.

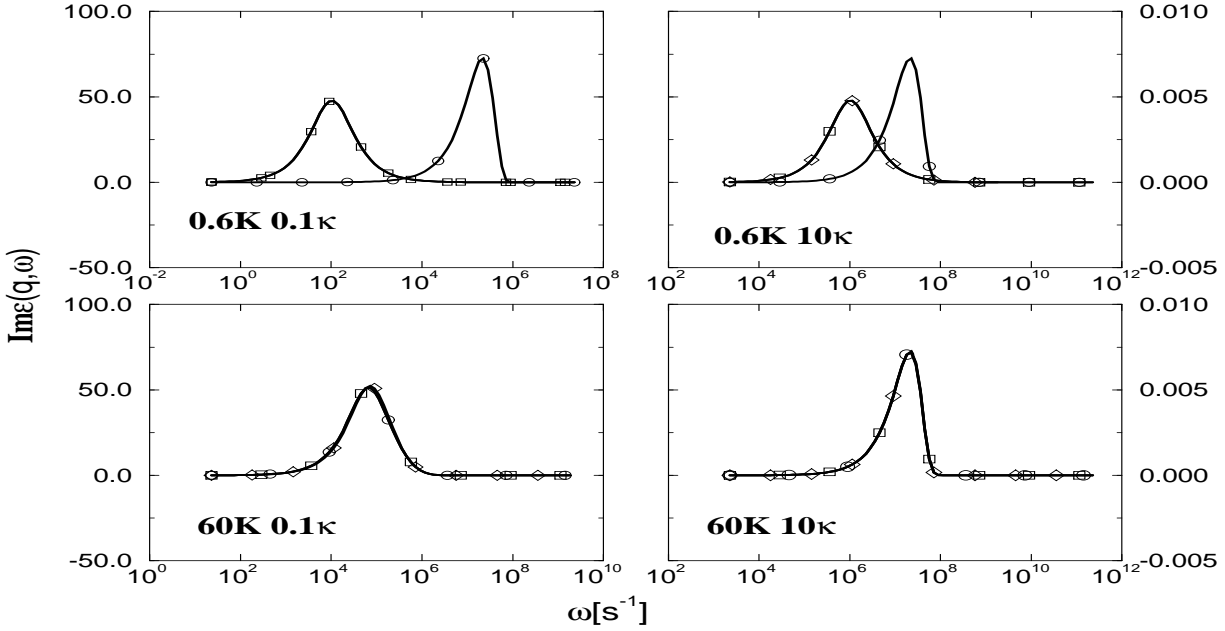


FIG. 2. Comparison of the imaginary parts of the RPA- (circles), Mermin- (squares) and VFP-DF (diamonds). The same density and temperatures are chosen as in the Fig. 1. The imaginary parts of Mermin- and the VFP-DF are identical.

C. Comparison of both dielectric function

Up to now we have used different kinetic equations leading to two different dielectric functions. It is of great interest how these functions are related to each other and whether these dielectric functions are valid in the storage ring plasma's realm of temperature, density and friction coefficient. In Fig.1 and Fig. 2 are plotted both dielectric functions in dependence on the frequency for several wave numbers and temperatures.

We see that within the numerical accuracy of the picture no difference is visible between the dielectric function of a Fokker Planck collision integral and the Mermin dielectric function resulting from the conserving relaxation time approximation.

Let us inspect now some special limits. Both dielectric function fulfills the static limit ($\omega \rightarrow 0$)

$$\epsilon(q, 0) = 1 + \frac{\kappa^2}{q^2} \quad (28)$$

for all λ in accordance with the classical Debye-Hückel result for static screening. In the long-wavelength limit $q \rightarrow 0$ one gets the Drude formula for both dielectric functions

$$\lim_{\substack{q \rightarrow 0 \\ \omega \rightarrow \infty}} \epsilon(q, \omega) = 1 - \frac{\omega_{pl}^2}{\omega(\omega + i\lambda)}. \quad (29)$$

For $\lambda \rightarrow 0$ this formula reproduces the RPA behavior. In the limit of strong friction $\lambda \rightarrow \infty$ we get in concur with [3] and [8] also (29). The long-wavelength and the strong friction limits are identical.

For low temperatures there are differences between the RPA-dielectric function and the other correlated dielectric functions. The real parts start in the static limit at the same value as the RPA-dielectric function but drops down much more earlier (in Fig.1 one sees 4 magnitudes at one tenth of the inverse Debye length and 2 magnitudes at ten times the inverse Debye length). There are no zeros in the real part. Accordingly, the imaginary part is shifted in the same fashion. It is one magnitude broader than the RPA imaginary part and has only two thirds of its height. For temperatures higher than 50K the RPA-dielectric function and the Mermin- and VFP dielectric functions become identical.

D. Sum rules

The most interesting question is whether the dielectric function fulfills the sum rules (8)-(11). Due to (29) all presented dielectric functions lead to $\text{Re}\epsilon(\vec{q}, \omega) \propto \omega^{-2}$ for large ω . Since poles due to the relaxation time occur only in the lower half plane we have

$$\int_{-\infty}^{\infty} d\bar{\omega} \frac{\epsilon^{-1}(\vec{q}, \bar{\omega}) - 1}{(\bar{\omega} + i\eta)} = 0 \quad (30)$$

from which we see that the dielectric functions fulfill the Kramers-Kronig relations

$$\text{Re}\epsilon^{-1}(\vec{q}, \omega) - 1 = \mathcal{P} \int_{-\infty}^{\infty} \frac{2 \text{Im}\epsilon^{-1}(\vec{q}, \bar{\omega})}{\omega - \bar{\omega}} \frac{d\bar{\omega}}{2\pi} \quad (31)$$

where the \mathcal{P} denotes here the Cauchy principle value.

From (31) we get with (28) in the static limit just the compressibility sum rule (10). The longitudinal f-sum rule (8) follows as well from (31). To see this we observe that due to time reversibility $\epsilon(q, -\omega) = \epsilon^*(q, \omega)$ holds and we can write

$$\lim_{\omega \rightarrow \infty} \text{Re}\epsilon^{-1}(q, \omega) = 1 + \lim_{\omega \rightarrow \infty} \frac{2}{\omega^2 \pi} \int_0^{\infty} d\bar{\omega} \text{Im}\epsilon^{-1}(q, \bar{\omega}). \quad (32)$$

Using (29) we obtain just the f-sum rule (8).

Since the same Kramers-Kronig relation (31) holds also for ϵ instead of ϵ^{-1} we see that the corresponding free sum rules (9) and (11) are also fulfilled.

This completes the proof that both correlated dielectric functions fulfill the sum rules. We can state therefore that the dielectric function is properly valid in the interesting scope and can be used to describe the phenomena in cold and dilute storage ring plasmas. Since the Mermin dielectric function is computational much easier to handle, in fact no more circumstances than the Lindhard dielectric function, we will use the Mermin dielectric function further on.

III. APPLICATION TO STORAGE RING PLASMAS

We continue now to apply the correlated dielectric function derived in the last paragraph to typical storage ring plasmas. Two important quantities we like to discuss here. First the stopping power of ions in an electron plasma and second the occurring plasmon excitations.

A. Stopping power

The stopping power, i.e. the energy transfer of a particle to a plasma is given in terms of the dielectric function by [9]

$$\frac{dE_a}{dt} = -\frac{2}{\hbar} \int \frac{d^3q}{(2\pi\hbar)^3} \hbar\omega n_B(\hbar\omega) V_{aa}(q)^2 \text{Im}\epsilon^{-1}(q, \hbar\omega). \quad (33)$$

Here n_b denotes the Bose function and V_{aa} the Coulomb potential of the particle a . We observe that the sum

about different plasma species is condensed in the dielectric function. It is noteworthy to remark that this result is valid for any arbitrary degeneracy. The derivation presented in [9] shows that the result (33) is more general valid than derived earlier [10–12]. Higher order correlations like vertex corrections can be incorporated in the dielectric function, such that (33) remains valid [9]. This fact is important for dense solid state plasmas which are used recently for stopping experiments, where the result (33) is applicable as well. A more explicit form can be given by carrying out the angular integration [$q = \hbar k$]

$$\frac{\partial E_a}{\partial t} = \frac{2e_a^2}{\pi\epsilon_0} \frac{1}{v(t)} \int_0^\infty \frac{dk}{k} \int_{-v(t)k + \frac{\hbar k^2}{2m_a}}^{v(t)k + \frac{\hbar k^2}{2m_a}} d\omega \omega n_B(\omega) \text{Im}\epsilon^{-1}(\hbar k, \omega). \quad (34)$$

Neglecting the quantum effects in (34) which represent an internal ionic cut-off due to the thermal De Broglie wavelength we get the standard result of dielectric theory

$$\frac{\partial E_a}{\partial t} = \frac{2e_a^2}{\pi\epsilon_0} \frac{1}{v(t)} \int_0^\infty \frac{dk}{k} \int_0^{v(t)k} d\omega \omega \text{Im}\epsilon^{-1}(\hbar k, \omega) \quad (35)$$

from which all known special cases can be derived [9], among them the well known Bethe formula. We use (34) where no artificial cut-off is needed further on.

In figure 3 we have plotted the stopping power of $^9\text{Be}^+$ ions calculated with the Mermin and the Lindhard dielectric function. We observe that for a weakly coupled storage ring plasma with temperature of 6 K and a density of $2.3 \cdot 10^{13} \text{m}^{-3}$ which corresponds to a nonideality of $\Gamma = 0.13$ almost no differences are observed between the Mermin and Lindhard result. For higher coupling by lower temperature of 1 K corresponding to $\Gamma = 0.77$ we see that the Mermin stopping power become smaller than the Lindhard result. Since the friction is dependent on the squared of density but only on temperature via the Coulomb logarithm we find a stronger dependence on the density. This is illustrated in the figures 3-5. We see that with increasing density the deviations between Mermin and Lindhard results become appreciable.

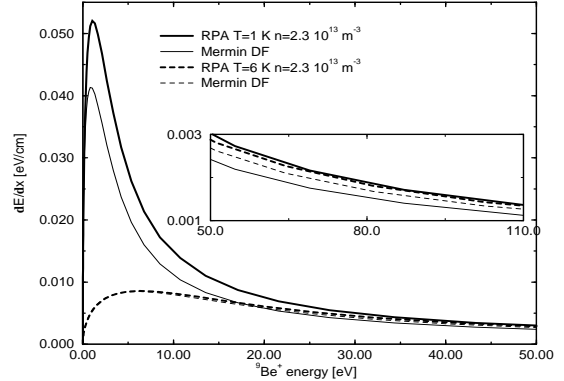


FIG. 3. The stopping power of $^9\text{Be}^+$ ions in an electronic plasma versus ion energy. The classical Lindhard result (thick lines) are compared with the Mermin result (thin lines) for two different temperatures. The plasma parameters are $\Gamma = 0.77$ (solid lines) and $\Gamma = 0.13$ (dashed lines) respectively.

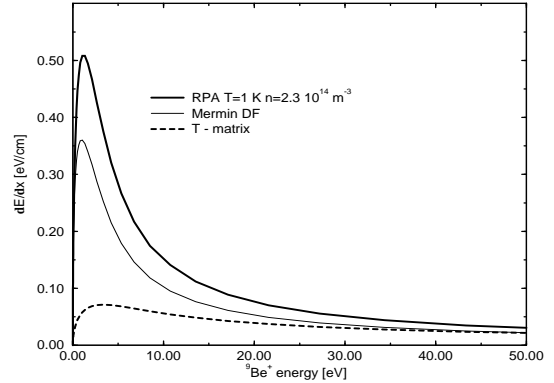


FIG. 4. The stopping power of $^9\text{Be}^+$ ions in an electronic plasma versus ion energy. The classical Lindhard result (thick line) is compared with the Mermin result (thin line) and the T-matrix result (dashed line) of binary collisions (36). The plasma parameter is $\Gamma = 1.65$.

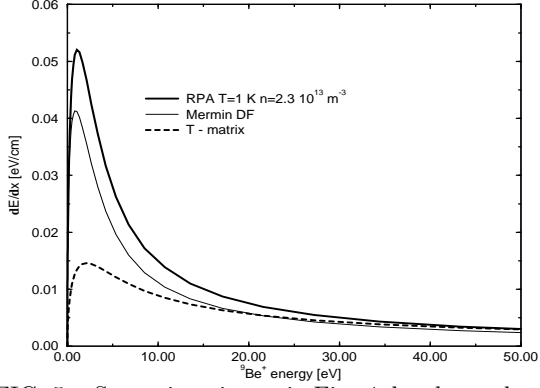


FIG. 5. Same situation as in Fig. 4, but lower density $\Gamma = 0.77$.

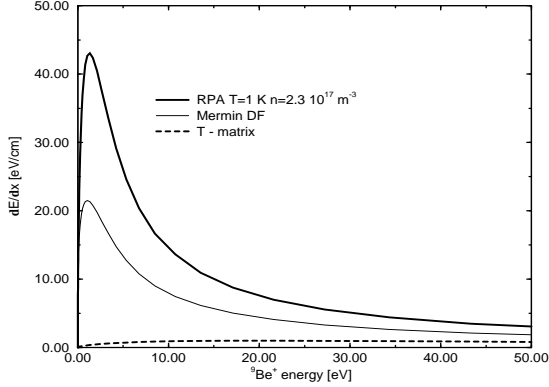


FIG. 6. Same as in Fig. 4, but density is now 4 magnitudes higher ($\Gamma = 16.5$)

So far we have generalized the dielectric theory of stopping power by the inclusion of collisions. It is instructive to compare now the results directly with the stopping power in binary collision approximation. In [9] the following expression for the stopping power was derived from the Boltzmann equation within T-matrix approximation

$$\frac{\partial E}{\partial t}(v) = \sum_b \frac{n_b v_t}{m_b^2 \sqrt{\pi}} \frac{e^{-\frac{m_b v^2}{2k_B T}}}{v} \int_0^\infty dp p^2 \sigma_{ab}^t(p) \left[a \cosh a - \left(1 + \frac{p^2(1 + \frac{m_b}{m_a})}{m_a k_B T}\right) \sinh a \right] e^{-\frac{p^2}{2m_b k_B T} (1 + \frac{m_b}{m_a})^2} \quad (36)$$

with the thermal velocity $v_t^2 = 2k_B T/m_b$, the abbreviation $a = \frac{vp}{k_B T} (1 + \frac{m_b}{m_a})$ and the quantum mechanical transport cross section

$$\sigma^t(p) = \int d\Omega (1 - \cos\theta) \frac{d\sigma}{d\Omega}. \quad (37)$$

In [13] a fit formula is given which subsumed the numer-

ical results for the transport cross section for a plasma with charge $Z = 1$. In figures 3 - 5 we compare the results for the dielectric theory of stopping power with and without collisional contributions with the pure two-particle collision result of (36). We see that the two-particle collision expression is significantly smaller than the dielectric theory. For very strong coupling in figure 6 we see even a vanishing contribution of the latter one indicating that the two-particle collisions do not contribute any more but the energy transfer is mainly caused by collective spreading. In Fig.7 we represent the reduced energy loss $[\lambda_I = e^2/(12\pi\epsilon_0 T)]$

$$\frac{\lambda_I v_{th}}{k_B T} \frac{1}{v} \frac{dE}{dx} \quad (38)$$

versus the coupling parameter $Z\Gamma^{3/2}$.

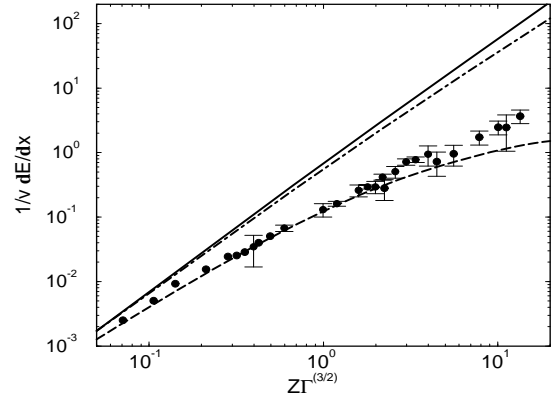


FIG. 7. The normalized friction coefficient (energy loss) for RPA(solid line), T-matrix result (dashed line) and Mermin DF result (dot-dashed line). The filled circles are simulation results [14] which reproduces experimental data [15,16].

The dependence of the normalized energy loss from the coupling parameter is weaker in the Mermin case than in the RPA case but distinct from the numerical simulations. Nevertheless the involving of collisions modifies the stopping power in the right direction. The best description is still given by the T-matrix result (36).

B. Plasmons

The ion beam current $j(t)$ is a fluctuating quantity due to its granular (ionic) structure. Detecting the mirror charge on the vacuum chamber of the ring and Fourier transforming (frequency analyzing) one obtains the Schottky-signal. It is primarily used for analyzing the beam's velocity distribution and hence the longitudinal temperature, but also important for measuring the particle number or the revolution frequency of the beam.

It is related to the dynamical structure factor $S(\vec{q}, \omega)$ by the equation [17]

$$\langle |J(q, \omega)|^2 \rangle \sim S(q, \omega), \quad (39)$$

where the brackets indicate the thermal averaging. The well-known fluctuation-dissipation theorem [18] connects the imaginary part of the response function $\text{Im}\epsilon(\vec{q}, \omega)$ and the dynamical structure factor

$$S(\vec{q}, \omega) = -\frac{k_B T}{\omega V_C(\vec{q})} \text{Im}\epsilon^{-1}(\vec{q}, \omega)$$

with the Coulomb potential V_C . In dense ion beams (e.g. the C^{6+} LIR experiment) one observes a double peaked Schottky spectrum. The two peaks commonly identified as plasma waves propagating in two directions around the storage ring. A frequency analysis of this beam current shows the propagating waves clearly as peaks in the spectra which are theoretically well described in [3,8]. The identity of Mermin and VFP-dielectric function we use now to compute the the Schottky noise much easier within the Mermin DF.

For numerical calculation one has to modify the plasma frequency which differs from the one of an isotropic plasma. For a plasma in a conducting tube we have

$$\tilde{\omega}_{pl}^2 = \frac{N Z^2 e^2 M}{2\pi R_0^3 m \epsilon_0 \gamma} \left(\ln \frac{r_C}{r_B} + \frac{1}{2} \right) \quad (40)$$

Here denotes N the particles number, r_C the radius of the beam chamber, r_B the radius of the beam propagating with the velocity $v_0 = \gamma c$ (speed of light), $2\pi R_0$ is the circumference of the ring. M is the number of plasma waves fitting in the ring, the so-called harmonic wavenumber.

The equation for the plasma frequency follows straightforward from inserting the beam pipe's impedance \mathcal{Z} via Ohm's law $U = e^2 v_0 \mathcal{Z} / R_0$ in the external field of the Vlasov equation. Usually \mathcal{Z} is given by

$$\mathcal{Z} = \frac{M}{v_0 \gamma^2} \left(\ln \left(\frac{r_C}{r_B} \right) + \frac{1}{2} \right). \quad (41)$$

We assume for simplicity the non-relativistic case which is valid in the TSR experiments ($^9\text{Be}^+$: $0.04c$, $^{12}\text{C}^{6+}$: $0.15c$) with $\gamma \approx 1$.

The wavelength q is not a continuous variable but assumes only discrete values $q = M/2\pi R_0$. The fraction κ^2/q^2 can now expressed by

$$\frac{\kappa^2}{q^2} = 2 \frac{\tilde{\omega}_{pl}^2}{\delta\omega^2} \quad (42)$$

with the thermal frequency

$$\delta\omega^2 = \frac{2k_B T}{R_0^2 m}. \quad (43)$$

Inserting this parameters into the VFP dielectric function one obtains

$$\epsilon_{\text{VFP}}(M, \omega) = 1 + 2 \left(\frac{\tilde{\omega}_{pl}}{\delta\omega} \right)^2 {}_1F_1 \left[1, 1 + \frac{M^2 \delta\omega^2}{2\lambda^2} - \frac{i\omega}{\lambda}, \frac{M^2 \delta\omega^2}{2\lambda^2} \right]. \quad (44)$$

This is the well known standard permittivity in the Schottky noise theory. Since the Mermin DF and the VFP-DF are identical we use practically now the easier Mermin DF. We modify equation (24) according to the parameters (40)-(43)

$$\epsilon_M(M, \omega + i\lambda) = 1 + \frac{(1 + \frac{\lambda}{\omega})(\epsilon(M, \omega + i\lambda) - 1)}{1 + \frac{i\lambda}{\omega} \frac{\epsilon(M, \omega + i\lambda) - 1}{\epsilon(M, 0) - 1}} \quad (45)$$

$$(46)$$

with (17) for $\epsilon(M, \omega + i\lambda)$ and

$$\epsilon_M(M, 0) = 1 + 2 \frac{\tilde{\omega}_{pl}^2}{\delta\omega^2}. \quad (47)$$

In the next step we insert a relaxation time considering the anisotropy in the thermal velocities v_{th} [5]

$$\lambda = 4\pi(Ze)^4 \frac{n}{\epsilon^2 m^2 v_{th,\perp}^2 v_{th,\parallel}^2} \Lambda. \quad (48)$$

Here Λ again denotes the Coulomb logarithm. We have used the modified Mermin-dielectric function for calculating the plasmonic excitation for a $^{12}\text{C}^{6+}$ beam at an energy of 73.3MeV and a revolution frequency of $v_0/R_0 = 617\text{kHz}$ ($M = 5$). In 8 the expression $\text{Im}\epsilon(M, \omega)/\omega$ is compared with the Schottky measurement. We see that the modified Mermin-dielectric function fits the Schottky spectra satisfactorily.

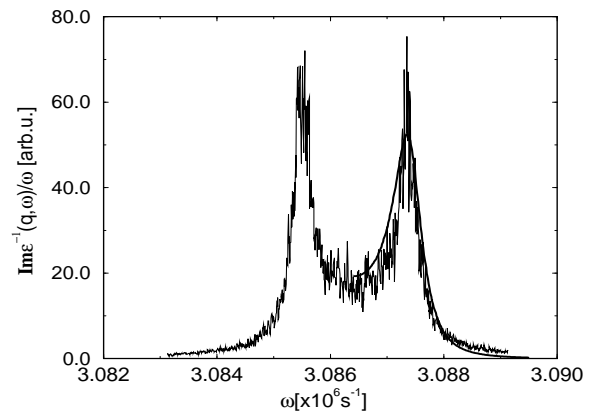


FIG. 8. The Schottky spectra of a dense carbonium beam and the corresponding theoretical prediction. ($n = 8.3 \cdot 10^{13} \text{m}^{-3}$, $T_{\parallel} = 11000\text{K}$). Data are taken from [19].

IV. SUMMARY

In this paper we have described two dielectric functions including collisions. After numerical inspection we have shown that the DF obtained from VFP equation is identical with the Mermin DF. Modifying the dielectric function for storage ring purposes (modified plasma frequency, discrete wavenumbers/harmonics) we have derived the standard dielectric function for Schottky noise prediction for a three dimensional plasma beam. Because of the identity of both dielectric functions one can use a Mermin dielectric function for Schottky noise description, too. The second goal was a better description of the stopping power acting on an ion beam in the cooler's electron gas. Here we like to state, that including the collisions leads to a lower friction force than in the RPA-predictions but obvious overestimates the friction force compared with simulated and experimental results.

Further improvement has to be done to consider strong coupling effects in the relaxation time by using a matching Coulomb logarithm. Efforts have also to be done to include the magnetic field in the cooler and the anisotropic Maxwell distributions in the dielectric function.

APPENDIX A: SOLUTION OF THE VLASOV-FOKKER-PLANCK EQUATION

We following the the main ideas of [20]. At first we introduce reduced variables

$$\vec{u} = \sqrt{\frac{m}{2k_B T}} \vec{v} \quad \vec{k} = \frac{\vec{q}}{\lambda} \sqrt{\frac{k_B T}{2m}}. \quad (\text{A1})$$

After linearization the VFP-equation (25) we arrive at

$$\begin{aligned} & \frac{\partial}{\partial t} \delta f(\vec{k}, \vec{u}, t) + 2i\lambda \vec{u} \vec{k} \delta f(\vec{k}, \vec{u}, t) - \frac{2i\lambda}{m} \vec{k} U(\vec{k}, t) \frac{\partial f_0(\vec{u})}{\partial \vec{u}} \\ & = \lambda \frac{\partial}{\partial \vec{u}} \left(\frac{1}{2} \frac{\partial}{\partial \vec{u}} + \vec{u} \right) \delta f(\vec{k}, \vec{u}, t). \end{aligned} \quad (\text{A2})$$

For the Fokker-Planck term including the second term of the left side (the so-called inhomogeneous Fokker-Planck operator) one can consider the eigenvalue and eigenfunctions in three dimensions

$$\begin{aligned} & \left[-2i\lambda \vec{u} \vec{k} + \lambda \frac{\partial}{\partial \vec{u}} \left(\frac{1}{2} \frac{\partial}{\partial \vec{u}} + \vec{u} \right) \right] \psi_{n_x n_y n_z}(\vec{k}, \vec{u}) \\ & = \Lambda_{n_x n_y n_z}(\vec{k}) \psi_{n_x n_y n_z}(\vec{k}, \vec{u}). \end{aligned} \quad (\text{A3})$$

The eigenfunctions take the form after a coordinate transformation $z = u + 2ik$

$$\begin{aligned} \psi_{n_x n_y n_z} & = \sqrt{\frac{1}{n_x! n_y! n_z! 2^{n_x+n_y+n_z}}} \left(\frac{m}{2\pi k_B T} \right)^3 \\ & \times e^{k^2} e^{-(z-i\vec{k})^2} H_{n_x}[z_x] H_{n_y}[z_y] H_{n_z}[z_z]. \end{aligned} \quad (\text{A4})$$

One can identifying the functions H_n as Hermite-Polynoms. The lowest eigenfunction for $n_x = n_y = n_z = 0$ is the Maxwell-distribution. The eigenvalues are

$$\Lambda_{n_x n_y n_z} = -\lambda(n_x + n_y + n_z + 2k^2). \quad (\text{A5})$$

In the next step we insert for the right side the eigenvalues and eigenfunctions of the inhomogeneous Fokker-Planck operator

$$\begin{aligned} & \frac{\partial}{\partial t} \delta f(\vec{k}, \vec{u}, t) - \frac{2i\lambda}{m} \vec{k} U(\vec{k}, t) \frac{\partial f_0(\vec{u})}{\partial \vec{u}} \\ & = \lambda \sum_{n_x n_y n_z} \mu_{n_x n_y n_z} \psi_{n_x n_y n_z} \\ & \text{with } \mu_n(\vec{k}) = \frac{\Lambda_n(\vec{q})}{\lambda} \end{aligned} \quad (\text{A6})$$

and expand the distribution function $\frac{\partial}{\partial \vec{u}} f_0$ and the distortion $\delta f(\vec{u}, \vec{k}, t)$ in a sum of eigenfunctions of the Fokker-Planck operator

$$\begin{aligned} & \vec{k} \frac{\partial}{\partial \vec{u}} f_0(\vec{u}) = \vec{k} \sum_{n_x n_y n_z} \vec{a}_{n_x n_y n_z} \psi_{n_x n_y n_z}(\vec{u}, \vec{k}) \\ & \delta f(\vec{u}, \vec{k}, t) = \sum_{n_x n_y n_z} c_{n_x n_y n_z} \psi_{n_x n_y n_z}. \end{aligned} \quad (\text{A7})$$

Performing a Fourier transformation $t \rightarrow \omega$ we arrive at a solution for the coefficients

$$c_{n_x n_y n_z} = \frac{\frac{i\lambda}{k_B T} \vec{k} \vec{a}_n U(\vec{k}, \omega)}{-i\omega - \lambda(n_x + n_y + n_z + 2k^2)}. \quad (\text{A8})$$

Remembering (15) we obtain

$$\begin{aligned} \delta n(\vec{k}, \omega) & = \sum_{n_x n_y n_z} c_{n_x n_y n_z} \int_{Vol} \psi_{n_x n_y n_z} d^3 u \\ & = U(\vec{k}, \omega) \Pi_{VFP}(\vec{k}, \omega). \end{aligned} \quad (\text{A9})$$

Hence we get for the polarizability

$$\begin{aligned} \Pi_{VFP}(k, \omega) & = \frac{\lambda}{2\pi k_B T} \exp[k^2] \sum_{n_x n_y n_z} i^{2(n_x+n_y+n_z)} \\ & \times \frac{\Gamma[\frac{1}{2} + n_x] \Gamma[\frac{1}{2} + n_y] \Gamma[\frac{1}{2} + n_z] \Gamma[\frac{1}{2} + n_x + n_y]}{n_x! n_y! n_z! (n_x + n_y)! (n_x + n_y + n_z)!} \\ & \times \frac{(n_x + n_y + n_z + 2k^2)(\sqrt{2}k)^{2(n_x+n_y+n_z)}}{-i\omega - \lambda(n_x + n_y + n_z + 2k^2)}. \end{aligned} \quad (\text{A10})$$

Using the relations

$$\begin{aligned} & \sum_{n_x, n_y, n_z} \frac{\Gamma[\frac{1}{2} + n_x] \Gamma[\frac{1}{2} + n_y] \Gamma[\frac{1}{2} + n_z] \Gamma[\frac{1}{2} + n_x + n_y]}{n_x! n_y! n_z! (n_x + n_y)! (n_x + n_y + n_z)!} \\ & = \sum_m \frac{\pi^2}{m!}, \end{aligned} \quad (\text{A11})$$

and

$$\sum_n \frac{(-x)^n}{n!(\kappa+n)} = \frac{\exp[-x]}{\kappa} {}_1F_1[1, 1+\kappa, x] \quad (\text{A12})$$

we arrive finally at

$$\Pi_{\text{VFP}}(k, \omega) = \frac{\pi}{2k_B T} \left(1 + \frac{i\omega}{2\lambda k^2 - i\omega} {}_1F_1 \left[1, 1 + 2k^2 - i\frac{\omega}{\lambda}; 2k^2 \right] \right). \quad (\text{A13})$$

The dielectric function related to this polarizability was discussed in chapter II B.

APPENDIX: ACKNOWLEDGMENTS

The authors acknowledge stimulating discussions with G. Röpke. This work was supported from the BMBF (Germany) under contract Nr. 06R0884 and the Max-Planck-Society.

-
- [1] D. Habs and R. Grimm, *Ann. Rev. Nucl. Part. Sci.* **45** 391 (1995)
 - [2] H. Poth, *Phys. Rep.* **196** 135 (1990)
 - [3] V. Parkhomchuk and D. Pestrikov, *Zh. Tech. Fis.* **50** 1411 (1980) [russ.] (*Sov. Phys. Tech. Phys.* **25** 818 (1980))
 - [4] U. Schramm, *Hyp. Int.* **108** 273 (1997)
 - [5] A. H. Sørensen in: *CERN Accelerator School*, Hrsg. S. Turner, CERN 87-10 (1987)
 - [6] G. Röpke and A. Wierling, *Phys. Rev.* **E57** 7075 (1998)
 - [7] N. D. Mermin, *Phys. Rev.* **B1** 2362 (1970)
 - [8] V. A. Lebedev, J. S. Hangst, J. S. Nielsen, *Phys. Rev.* **E52** 4345 (1995)
 - [9] K. Morawetz and G. Röpke, *Phys. Rev.* **E54**, 1 (1996).
 - [10] W. D. Kraeft and B. Strege, *Physica* **A149**, 313 (1988).
 - [11] H. H. Brouwer *et al.*, *Contrib. Plasma Phys.* **30**, 263 (1990).
 - [12] H. H. Brouwer *et al.*, *Contrib. Plasma Phys.* **30**, 369 (1990).
 - [13] G. Röpke and R. Redmer, *Phys. Rev. A* **39**, 907 (1989).
 - [14] G. Zwicknagel, C. Toepffer, P. Reinhard, *Hyp. Int.* **99** 285 (1996)
 - [15] A. Wolf *et al.*, *Proc. Workshop on Beam Cooling and Related Topics*, Montreaux, 1993 **CERN 94-03** Geneva, 1994, 416
 - [16] T. Winkler *et al.*, *Hyp. Int.* **99** 277 (1996)
 - [17] V. Avilov, I. Hofmann, *Phys. Rev.* **E47** 2019 (1993)
 - [18] R. Kubo, *J. Phys. Soc. Jap.* **12** (1957)
 - [19] K. Tetzlaff, diploma thesis **MPI-HV23-1997**
 - [20] P. Resibois, *Classical Kinetics - Theory of Fluids*. New York 1996